

Axicon Gaussian Laser Beams

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(March 14, 2000)

1 Problem

Deduce an axicon solution for a Gaussian laser beam in vacuum, *i.e.*, a beam with radial polarization of the electric field.

2 Solution

If a laser beam is to have radial transverse polarization, the transverse electric must vanish on the symmetry axis, which is charge free in vacuum. However, we can expect a nonzero longitudinal electric field on the axis, noting that the projections onto the axis of the electric field vectors of rays all have the same sign, as shown in Fig. 1a. This contrasts with the case of linearly polarized Gaussian laser beams [2, 3, 4, 5] for which rays at 0° and 180° azimuth to the polarization direction have axial electric field components of opposite sign, as shown in Fig. 1b. The longitudinal electric field of axicon laser beams may be able to transfer net energy to charged particles that propagate along the optical axis, providing a form of laser acceleration [6, 7, 8, 9].

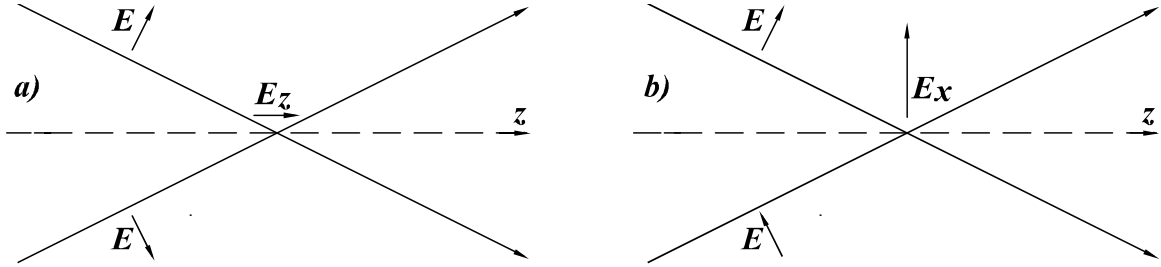


Figure 1: a) The radial polarization of the electric field of an axicon laser beam leads to a longitudinal electric field at the focus. b) For a linearly polarized laser beam, shown here with polarization along the x axis, the electric field is transverse at the focus.

Although two of the earliest papers on Gaussian laser beams [10, 11] discuss axicon modes (without using that term, and without deducing the simplest axicon mode), most subsequent literature has emphasized linearly polarized Gaussian beams. We demonstrate that a calculation that begins with the vector potential (sec. 2.1) leads to both the lowest-order linearly polarized and axicon modes. We include a discussion of Gaussian laser pulses as well as continuous beams, and find in sec. 2.2 that the temporal pulse shape must obey condition (8). The paraxial wave equation and its lowest-order, linearly polarized solutions are reviewed in secs. 2.3-4. Readers familiar with the paraxial wave equation for linearly

polarized Gaussian beams may wish to skip directly to sec. 2.5 in which the axicon mode is displayed. In sec. 2.6 we find an expression for a guided axicon beam, *i.e.*, one that requires a conductor along the optical axis.

2.1 Solution via the Vector Potential

Many discussions of Gaussian laser beams emphasize a single electric field component such as $E_x = f(r, z)e^{i(kz - \omega t)}$ of a cylindrically symmetric beam of angular frequency ω and wave number $k = \omega/c$ propagating in vacuum along the z axis. Of course, the electric field must satisfy the free-space Maxwell equation $\nabla \cdot \mathbf{E} = 0$. If $f(r, z)$ is not constant and $E_y = 0$, then we must have nonzero E_z . That is, the desired electric field has more than one vector component.

To deduce all components of the electric and magnetic fields of a Gaussian laser beam from a single scalar wave function, we follow the suggestion of Davis [12] and seek solutions for a vector potential \mathbf{A} that has only a single component. We work in the Lorentz gauge (and Gaussian units), so that the scalar potential Φ is related to the vector potential by

$$\nabla \cdot \mathbf{A} + \frac{1}{c} \frac{\partial \Phi}{\partial t} = 0. \quad (1)$$

The vector potential can therefore have a nonzero divergence, which permits solutions having only a single component. Of course, the electric and magnetic fields can be deduced from the potentials via

$$\mathbf{E} = -\nabla \Phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}, \quad (2)$$

and

$$\mathbf{B} = \nabla \times \mathbf{A}. \quad (3)$$

For this, the scalar potential must first be deduced from the vector potential using the Lorentz condition (1).

The vector potential satisfies the free-space wave equation,

$$\nabla^2 \mathbf{A} = \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2}. \quad (4)$$

We seek a solution in which the vector potential is described by a single component A_j that propagates in the $+z$ direction with the form

$$A_j(\mathbf{r}, t) = \psi(r_\perp, z)g(\varphi)e^{i\varphi}, \quad (5)$$

where the spatial envelope ψ is azimuthally symmetric, $r_\perp = \sqrt{x^2 + y^2}$, g is the temporal pulse shape, and the phase φ is given by

$$\varphi = kz - \omega t. \quad (6)$$

Inserting trial solution (5) into the wave equation (4) we find that

$$\nabla^2 \psi + 2ik \frac{\partial \psi}{\partial z} \left(1 - \frac{ig'}{g}\right) = 0, \quad (7)$$

where $g' = dg/d\varphi$.

2.2 A Condition on the Temporal Pulse Shape $g(\varphi)$

Since ψ is a function of \mathbf{r} while g and g' are functions of the phase φ , eq. (7) cannot be satisfied in general. Often the discussion is restricted to the case where $g' = 0$, *i.e.*, to continuous waves. For a pulsed laser beam, g must obey

$$\left| \frac{g'}{g} \right| \ll 1 \quad (8)$$

for eq. (7) to be consistent.

It is noteworthy that a ‘‘Gaussian’’ laser beam cannot have a Gaussian temporal pulse. That is, if $g = \exp[-(\varphi/\varphi_0)^2]$, then $|g'/g| = 2|\varphi|/\varphi_0^2$, which does not satisfy condition (8) for $|\varphi|$ large compared to the characteristic pulsewidth $\varphi_0 = \omega\Delta t$, *i.e.*, in the tails of the pulse.

A more appropriate form for a pulsed beam is a hyperbolic secant (as arises in studies of solitons):

$$g(\varphi) = \text{sech} \left(\frac{\varphi}{\varphi_0} \right). \quad (9)$$

Then, $|g'/g| = (1/\varphi_0) |\tanh(\varphi/\varphi_0)|$, which is less than one everywhere provided that $\varphi_0 \gg 1$.

2.3 The Paraxial Wave Equation

In the remainder of this paper, we suppose that condition (8) is satisfied. Then, the differential equation (7) for the spatial envelope function ψ becomes

$$\nabla^2 \psi + 2ik \frac{\partial \psi}{\partial z} = 0. \quad (10)$$

The function ψ can and should be expressed in terms of three geometric parameters of a focused beam, the diffraction angle θ_0 , the waist w_0 , and the depth of focus (Rayleigh range) z_0 , which are related by

$$\theta_0 = \frac{w_0}{z_0} = \frac{2}{kw_0}, \quad \text{and} \quad z_0 = \frac{kw_0^2}{2} = \frac{2}{k\theta_0^2}. \quad (11)$$

We therefore work in the scaled coordinates

$$\xi = \frac{x}{w_0}, \quad v = \frac{y}{w_0}, \quad \rho^2 = \frac{r_{\perp}^2}{w_0^2} = \xi^2 + v^2, \quad \text{and} \quad \varsigma = \frac{z}{z_0}, \quad (12)$$

Changing variables and noting relations (11), eq. (10) takes the form

$$\nabla_{\perp}^2 \psi + 4i \frac{\partial \psi}{\partial \varsigma} + \theta_0^2 \frac{\partial^2 \psi}{\partial \varsigma^2} = 0, \quad (13)$$

where

$$\nabla_{\perp}^2 \psi = \frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial v^2} = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \psi}{\partial \rho} \right), \quad (14)$$

since ψ is independent of the azimuth ϕ .

The form of eq. (13) suggests the series expansion

$$\psi = \psi_0 + \theta_0^2 \psi_2 + \theta_0^4 \psi_4 + \dots \quad (15)$$

in terms of the small parameter θ_0^2 . Inserting this into eq. (13) and collecting terms of order θ_0^0 and θ_0^2 , we find

$$\nabla_{\perp}^2 \psi_0 + 4i \frac{\partial \psi_0}{\partial \zeta} = 0, \quad (16)$$

and

$$\nabla_{\perp}^2 \psi_2 + 4i \frac{\partial \psi_2}{\partial \zeta} = -\frac{\partial^2 \psi_0}{\partial \zeta^2}, \quad (17)$$

etc.

Equation (16) is called the the paraxial wave equation, whose solution is well-known to be

$$\psi_0 = f e^{-f \rho^2}, \quad (18)$$

where

$$f = \frac{1}{1 + i\zeta} = \frac{1 - i\zeta}{1 + \zeta^2} = \frac{e^{-i \tan^{-1} \zeta}}{\sqrt{1 + \zeta^2}}. \quad (19)$$

The factor $e^{-i \tan^{-1} \zeta}$ in f is the so-called Guoy phase shift [2], which changes from 0 to $\pi/2$ as z varies from 0 to ∞ , with the most rapid change near the z_0 .

The solution to eq. (17) for ψ_2 has been given in [12], and that for ψ_4 has been discussed in [13].

With the lowest-order spatial function ψ_0 in hand, we are nearly ready to display the electric and magnetic fields of the corresponding Gaussian beams. But first, we need the scalar potential Φ , which we suppose has the form

$$\Phi(\mathbf{r}, t) = \Phi(\mathbf{r}) g(\varphi) e^{i\varphi}, \quad (20)$$

similar to that of the vector potential. Then,

$$\frac{\partial \Phi}{\partial t} = -\omega \Phi \left(1 - \frac{ig'}{g} \right) \approx -\omega \Phi, \quad (21)$$

assuming condition (8) to be satisfied. In that case,

$$\Phi = -\frac{i}{k} \nabla \cdot \mathbf{A}, \quad (22)$$

according to the Lorentz condition (1). The electric field is then given by

$$\mathbf{E} = -\nabla \Phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \approx ik \left[\mathbf{A} + \frac{1}{k^2} \nabla (\nabla \cdot \mathbf{A}) \right], \quad (23)$$

in view of condition (8). Note that $(1/k) \partial / \partial x = (\theta_0/2) \partial / \partial \xi$, *etc.*, according to eqs. (11)-(12).

2.4 Linearly Polarized Gaussian Beams

Taking the scalar wave function (18) to be the x component of the vector potential,

$$A_x = \frac{E_0}{ik} \psi_0 g(\varphi) e^{i\varphi}, \quad A_y = A_z = 0, \quad (24)$$

the corresponding electric and magnetic fields are found from eqs. (3), (23) and (24) to be the familiar forms of a linearly polarized Gaussian beam,

$$\begin{aligned} E_x &= E_0 \psi_0 g e^{i\varphi} + \mathcal{O}(\theta_0^2) \approx E_0 f e^{-f\rho^2} g e^{i\varphi} \\ &= \frac{E_0 e^{-\rho^2/(1+\varsigma^2)} g(\varphi)}{\sqrt{1+\varsigma^2}} e^{i[kz+\varsigma\rho^2/(1+\varsigma^2)-\omega t-\tan^{-1}\varsigma]}, \\ &= \frac{E_0 e^{-r_\perp^2/w^2(z)} g(\varphi)}{\sqrt{1+z^2/z_0^2}} e^{i\{kz[1+r_\perp^2/2(z^2+z_0^2)]-\omega t-\tan^{-1}(z/z_0)\}}, \\ E_y &= 0, \\ E_z &= \frac{i\theta_0 E_0}{2} \frac{\partial \psi_0}{\partial \xi} g e^{i\varphi} + \mathcal{O}(\theta_0^3) \approx -i\theta_0 f \xi E_x, \end{aligned} \quad (25)$$

$$\begin{aligned} B_x &= 0, \\ B_y &= E_x, \\ B_z &= \frac{i\theta_0 E_0}{2} \frac{\partial \psi_0}{\partial v} g e^{i\varphi} = -i\theta_0 f v E_x, \end{aligned} \quad (26)$$

where

$$w(z) = w_0 \sqrt{1 + z^2/z_0^2} \quad (27)$$

is the characteristic transverse size of the beam at position z . Near the focus ($r_\perp \lesssim w_0$, $|z| < z_0$), the beam is a plane wave,

$$E_x \approx E_0 e^{-r_\perp^2/w_0^2} e^{i(kz-\omega t-z/z_0)}, \quad E_z \approx \theta_0 \frac{x}{w_0} E_0 e^{-r_\perp^2/w_0^2} e^{i(kz-\omega t-2z/z_0-\pi/2)}, \quad (28)$$

For large z ,

$$E_x \approx E_0 e^{-\theta^2/\theta_0^2} \frac{e^{i(kr-\omega t-\pi/2)}}{r}, \quad E_z \approx -\frac{x}{r} E_x, \quad (29)$$

where $r = \sqrt{r_\perp^2 + z^2}$ and $\theta \approx r_\perp/r$, which describes a linearly polarized spherical wave of extent θ_0 about the z axis. The fields E_x and E_z , *i.e.*, the real parts of eqs. (29), are shown in Figs. 2 and 3.

The fields (25)-(26) satisfy $\nabla \cdot \mathbf{E} = 0 = \nabla \cdot \mathbf{B}$ plus terms of order θ_0^2 .

Clearly, a vector potential with only a y component of form similar to eq. (24) leads to the lowest-order Gaussian beam with linear polarization in the y direction.

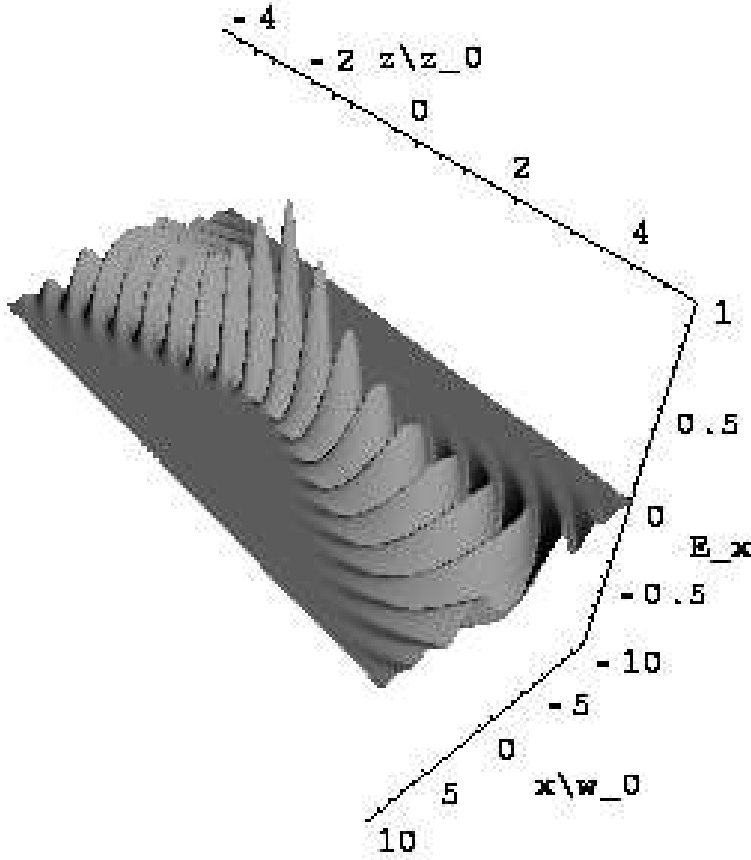


Figure 2: The electric field $E_x(x, 0, z)$ of a linearly polarized Gaussian beam with diffraction angle $\theta_0 = 0.45$, according to eq. (27).

2.5 The Lowest-Order Axicon Beam

An advantage of our solution based on the vector potential is that we also can consider the case that only A_z is nonzero and has the form (18),

$$A_x = A_y = 0, \quad A_z = \frac{E_0}{k\theta_0} f e^{-f\rho^2} g e^{i(kz - \omega t)}. \quad (30)$$

Then,

$$\nabla \cdot \mathbf{A} = \frac{\partial A_z}{\partial z} \approx ikA_z \left[1 - \frac{\theta_0^2}{2} f(1 - f\rho^2) \right], \quad (31)$$

using eqs. (11)-(12) and the fact that $df/d\zeta = -if^2$, which follows from eq. (19). Anticipating that the electric field has radial polarization, we work in cylindrical coordinates, (r_\perp, ϕ, z) , and find from eqs. (3), (23), (30) and (31) that

$$\begin{aligned} E_\perp &= E_0 \rho f^2 e^{-f\rho^2} g e^{i\varphi} + \mathcal{O}(\theta_0^2), \\ E_\phi &= 0, \\ E_z &= i\theta_0 E_0 f^2 (1 - f\rho^2) e^{-f\rho^2} g e^{i\varphi} + \mathcal{O}(\theta_0^3). \end{aligned} \quad (32)$$

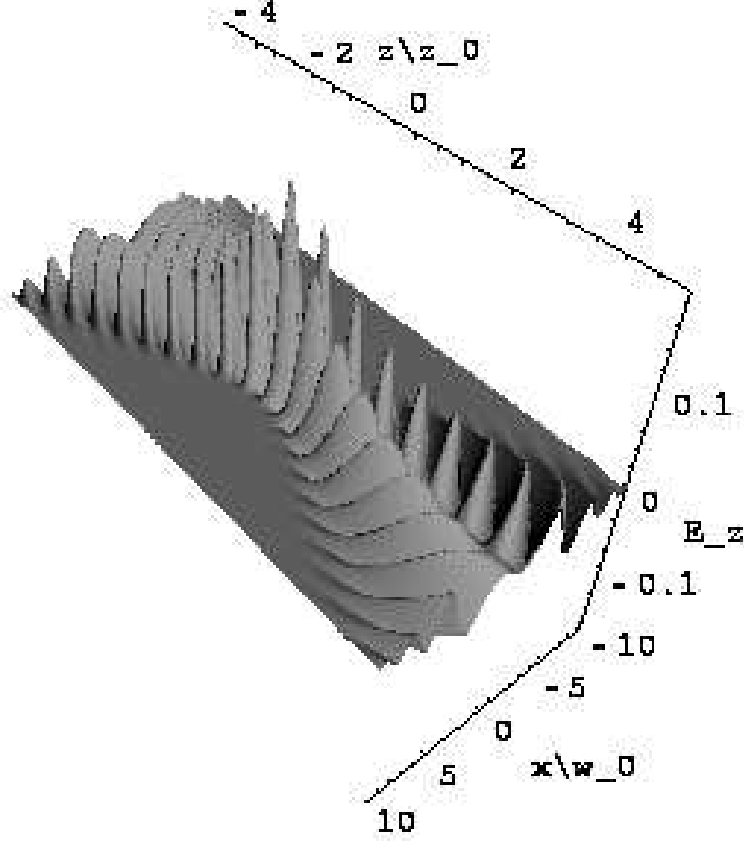


Figure 3: The electric field $E_z(x, 0, z)$ of a linearly polarized Gaussian beam with diffraction angle $\theta_0 = 0.45$, according to eq. (27).

The magnetic field is

$$B_{\perp} = 0, \quad B_{\phi} = E_{\perp}, \quad B_z = 0. \quad (33)$$

The fields E_x and E_z are shown in Figs. 4 and 5. The dislocation seen in Fig. 5 for $\rho \approx \varsigma$ is due to the factor $1 - f\rho^2$ that arises in the paraxial approximation, and would, I believe, be smoothed out on keeping higher-order terms in the expansion (15).

The transverse electric field is radially polarized and vanishes on the axis. The longitudinal electric field is nonzero on the axis. Near the focus, $E_z \approx i\theta_0 E_0$ and the peak radial field is $E_0/\sqrt{2e} = 0.42E_0$. For large z , E_{\perp} peaks at $\rho = \varsigma/\sqrt{2}$, corresponding to polar angle $\theta = \theta_0/\sqrt{2}$. For angles near this, $|E_{\perp}| \approx \rho|f|^2 \approx 1/z$, as expected in the far zone. In this region, the ratio of the longitudinal to transverse fields is $E_z/E_{\perp} \approx -i\theta_0 f\rho \approx -r_{\perp}/z$, as expected for a spherical wave front.

The factor f^2 in the fields implies a Guoy phase shift of $e^{-2i \tan^{-1} \varsigma}$, which is twice that of the lowest-order linearly polarized beams.

It is noteworthy that the simplest axicon mode (32)-(33) is not a member of the set of Gaussian modes based on Laguerre polynomials in cylindrical coordinates (see, for example, sec. 3.3b of [1]).

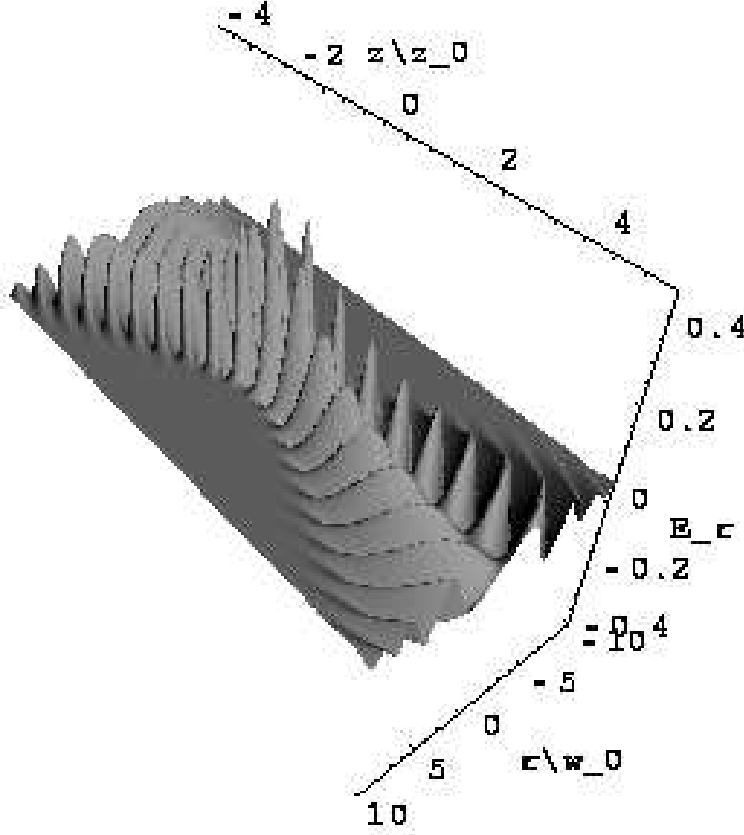


Figure 4: The electric field $E_r(r_\perp, 0, z)$ of an axicon Gaussian beam with diffraction angle $\theta_0 = 0.45$, according to eq. (32).

2.6 Guided Axicon Beam

We could also consider the vector potential

$$A_{r_\perp} \propto \psi_0 g e^{i\varphi}, \quad A_\phi = A_z = 0, \quad (34)$$

which leads to the electric and magnetic fields

$$E_r = E_0 f e^{-f\rho^2} g e^{i\varphi}, \quad E_\phi = 0, \quad E_z = -i\theta_0 f \rho E_r, \quad B_r = 0, \quad B_\phi = E_r, \quad B_z = 0, \quad (35)$$

and the potential

$$A_{r_\perp} = 0, \quad A_\phi \propto \psi_0 g e^{i\varphi}, \quad A_z = 0, \quad (36)$$

which leads to

$$E_r = 0, \quad E_\phi = E_0 f e^{-f\rho^2} g e^{i\varphi}, \quad E_z = 0, \quad B_r = -E_\phi, \quad B_\phi = 0, \quad B_z = -i\theta_0 \frac{1 - 2f\rho^2}{2\rho} E_\phi. \quad (37)$$

The case of eqs. (36)-(37) is unphysical due to the blowup of B_z as $r_\perp \rightarrow 0$.

The fields of eqs. (34)-(35) do not satisfy $\nabla \cdot \mathbf{E} = 0$ at $r_\perp = 0$, and so cannot correspond to a free-space wave. However, these fields could be supported by a wire, and represent a

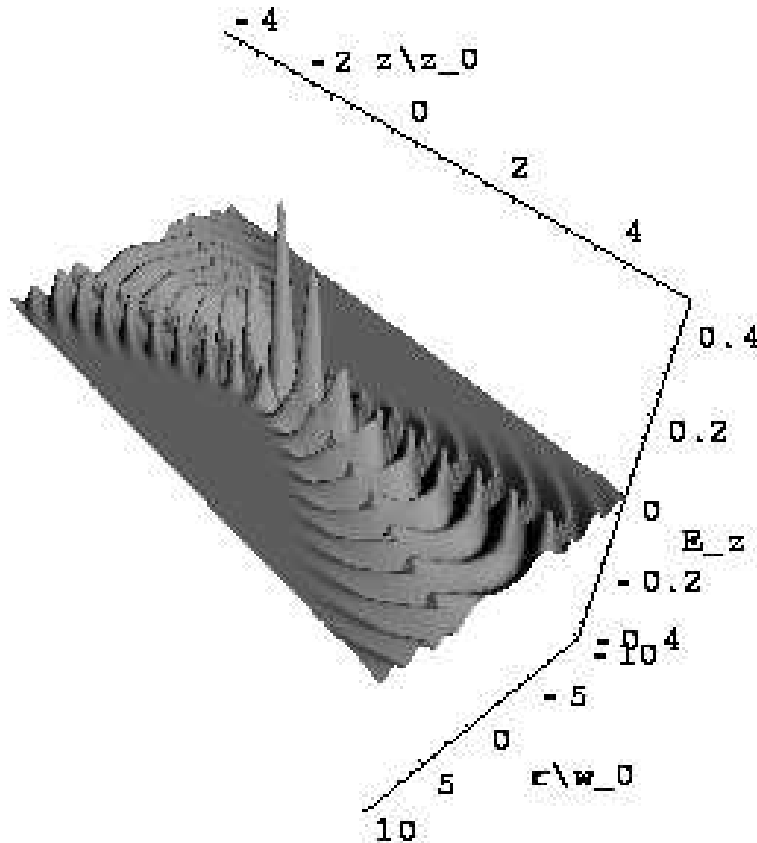


Figure 5: The electric field $E_z(r_\perp, 0, z)$ of an axicon Gaussian beam with diffraction angle $\theta_0 = 0.45$, according to eq. (32).

TM axicon guided cylindrical wave with a focal point. This is in contrast to guided plane waves whose radial profile is independent of z [14, 15]. Guided axicon beams might find application when a focused beam is desired at a point where a system of lenses and mirrors cannot conveniently deliver the optical axis, or in wire-guided atomic traps [16]. Figures 2 and 3 show the functional form of the guided axicon beam (35), when coordinate x is reinterpreted as r_\perp .

3 References

- [1] H. Kogelnik and T. Li, *Laser Beams and Resonators*, Appl. Opt. **5**, 1550-1567 (1966).
- [2] A.E. Siegman, *Lasers* (University Science Books, Mill Valley, CA, 1986), chaps. 16-17.
- [3] P.W. Milonni and J.H. Eberly, *Lasers* (Wiley Interscience, New York, 1988), sec. 14.5.
- [4] A. Yariv, *Quantum Electronics*, 3rd ed. (Wiley, New York, 1989), chap. 6.
- [5] K.T. McDonald, *Time Reversed Diffraction* (Sept. 5, 1999).

- [6] J.A. Edighoffer and R.H. Pantell, *Energy exchange between free electrons and light in vacuum*, J. Appl. Phys. **50**, 6120-6122 (1979).
- [7] F. Caspers and E. Jensen, *Particle Acceleration with the Axial Electric Field of a TEM_{10} Mode Laser Beam*, in *Laser Interaction and Related Plasma Phenomena*, ed. by H. Hora and G.H. Miley (Plenum Press, New York, 1991), Vol. 9, pp. 459-466.
- [8] E.J. Bochove, G.T. Moore and M.O. Scully, *Acceleration of particles by an asymmetric Hermite-Gaussian laser beam*, Phys. Rev. A **46**, 6640-6653 (1992).
- [9] L.C. Steinhauer and W.D. Kimura, *A new approach to laser particle acceleration in vacuum*, J. Appl. Phys. **72**, 3237-3245 (1992).
- [10] G. Goubau and F. Schwering, *On the Guided Propagation of Electromagnetic Wave Beams*, IRE Trans. Antennas and Propagation, **AP-9**, 248-256 (1961).
- [11] G.D. Boyd and J.P. Gordon, *Confocal Multimode Resonator for Millimeter Through Optical Wavelength Masers*, Bell Sys. Tech. J. **40**, 489-509 (1961).
- [12] L.W. Davis, *Theory of electromagnetic beams*, Phys. Rev. A **19**, 1177-1179 (1979).
- [13] J.P. Barton and D.R. Alexander, *Fifth-order corrected electromagnetic field components for a fundamental Gaussian beam*, J. Appl. Phys. **66**, 2800-2802 (1989).
- [14] A. Sommerfeld, *Electrodynamics* (Academic Press, New York, 1952), secs. 22-23.
- [15] J.A. Stratton, *Electromagnetic Theory* (McGraw-Hill, New York, 1941), secs. 9.16-17.
- [16] J. Denschlag, D. Cassettari and J. Schmiedmayer, *Guiding neutral atoms with a wire*, Phys. Rev. Lett. **82**, 2014-2017 (1999).